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2007 J. Phys. A: Math. Theor. 40 7187

(http://iopscience.iop.org/1751-8121/40/26/006)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 7187–7192 doi:10.1088/1751-8113/40/26/006

On the dynamics of the Bianchi IX system

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Received 10 November 2006, in final form 2 May 2007
Published 12 June 2007
Online at stacks.iop.org/JPhysA/40/7187

Abstract

In this paper, we study the flow on three invariant sets of dimension five for the classical Bianchi IX system. In these invariant sets, using the Darboux theory of integrability, we prove the non-existence of periodic solutions and we study their dynamics. Moreover, we find three invariant sets of dimension four where the flow is integrable.

PACS numbers: 02.30.Hq, 02.30.Ik

1. Introduction

The Bianchi IX model, also known as the mixmaster universe model, is obtained through a convenient solution of Einstein's equations and corresponds to the Hamiltonian system

$$\begin{aligned}
\dot q_1 &= 12q_1(p_1q_1 - p_2q_2 - p_3q_3),\\
\dot q_2 &= 12q_2(-p_1q_1 + p_2q_2 - p_3q_3),\\
\dot q_3 &= 12q_3(-p_1q_1 - p_2q_2 + p_3q_3),\\
\dot p_1 &= -12p_1(p_1q_1 - p_2q_2 - p_3q_3) - \tfrac{1}{3}(q_1 - q_2 - q_3),\\
\dot p_2 &= -12p_2(-p_1q_1 + p_2q_2 - p_3q_3) - \tfrac{1}{3}(-q_1 + q_2 - q_3),\\
\dot p_3 &= -12p_3(-p_1q_1 - p_2q_2 + p_3q_3) - \tfrac{1}{3}(-q_1 - q_2 + q_3),
\end{aligned} \tag{1}$$

in $\mathbb{R}^6$ with three degrees of freedom and with zero energy, i.e. $G = 0$ for the Hamiltonian

$$\begin{aligned}
G = {}& 6\big(p_1^2q_1^2 + p_2^2q_2^2 + p_3^2q_3^2 - 2p_1q_1p_2q_2 + 2p_1q_1p_3q_3 - 2p_2q_2p_3q_3\big)\\
&+ \tfrac{1}{6}\big(q_1^2 + q_2^2 + q_3^2 - 2q_1q_2 - 2q_1q_3 - 2q_2q_3\big).
\end{aligned} \tag{2}$$

Of course $\dot q_i = G_{p_i}$ and $\dot p_i = -G_{q_i}$ for $i = 1, 2, 3$. The function G is a *first integral* of system (1), i.e., it is a function which is constant over the trajectories of this system. As usual the dots in system (1) denote the derivative with respect to the time t.

This model has attracted the interest of both cosmologists and integrability specialists. See for instance [2, 4–7, 9–12].

Consider the coordinate change defined by

$$y_i = q_i, \qquad z_i = p_i q_i,$$

for $i = 1, 2, 3$. In these coordinates, system (1) becomes the following homogeneous polynomial differential system of degree 2 in $\mathbb{R}^6$:

$$\begin{aligned}
\dot{y}_1 &= y_1(z_1 - z_2 - z_3),\\
\dot{y}_2 &= y_2(-z_1 + z_2 - z_3),\\
\dot{y}_3 &= y_3(-z_1 - z_2 + z_3),\\
\dot{z}_1 &= -y_1(y_1 - y_2 - y_3),\\
\dot{z}_2 &= -y_2(-y_1 + y_2 - y_3),\\
\dot{z}_3 &= -y_3(-y_1 - y_2 + y_3),
\end{aligned} \tag{3}$$

and the first integral G can be written now as

$$\begin{aligned}
H = {} & \left(z_1^2 + z_2^2 + z_3^2 - 2z_1z_2 - 2z_1z_3 - 2z_2z_3\right)/2\\
& + \left(y_1^2 + y_2^2 + y_3^2 - 2y_1y_2 - 2y_1y_3 - 2y_2y_3\right)/2.
\end{aligned} \tag{4}$$

For $i = 1, 2, 3$, the hyperplane $y_i = 0$ is invariant by the flow of system (3) and z_i is a first integral on the hyperplane $y_i = 0$. Here, an *invariant set* under the flow of system (3) means that if an orbit of this system has a point on this set, then the whole orbit is contained into the set.

We observe that the equations of system (3) are invariant by the permutation

$$(y_1, y_2, y_3, z_1, z_2, z_3) \mapsto (y_2, y_3, y_1, z_2, z_3, z_1) \mapsto (y_3, y_1, y_2, z_3, z_1, z_2). \tag{5}$$

Therefore, to know the dynamics on the hyperplane $y_1 = 0$ is equivalent to know it at any hyperplane $y_i = 0$ for $i = 1, 2, 3$. Hence, in what follows we only study the dynamics on the hyperplane $y_1 = 0$ and only state explicitly the results for this hyperplane.

For every $c \in \mathbb{R}$, the solutions of system (3) restricted to the invariant four-dimensional hyperplane of codimension 2

$$\Delta = \{(y_1 = 0, y_2, y_3, z_1 = c, z_2, z_3) \in \mathbb{R}^6\}$$

are given by the solutions of the system

$$\begin{aligned}
\dot{y}_2 &= y_2(z_2 - z_3 - c),\\
\dot{y}_3 &= y_3(-z_2 + z_3 - c),\\
\dot{y}_4 &= -y_2(y_2 - y_3),\\
\dot{y}_5 &= -y_3(-y_2 + y_3).
\end{aligned} \tag{6}$$

Let $s < 4$. The functions $F_1, \ldots, F_s : \Delta \to \mathbb{R}$ are *independent* if the $s \times 4$ matrix

$$\frac{\partial(F_1, \ldots, F_s)}{\partial(y_2, y_3, z_2, z_3)}$$

has rank s at all points $(0, y_2, y_3, c, z_2, z_3) \in \Delta$, except perhaps on a subset of Δ of Lebesgue measure zero.

Our main results are the following three theorems.

Theorem 1. *If $c = 0$, then system (6) is integrable (i.e. it has three independent first integrals). Moreover, we provide the explicit expression of its solutions.*

We observe here that if a system is integrable, then we can obtain its orbits simply performing the intersections of the level sets of its first integrals.

Let $\varphi(t) = \varphi(t, p)$ be the solution of system (6) passing through the point $p = (0, \overline{y_2}, \overline{y_3}, c, \overline{z_2}, \overline{z_3}) \in \Delta$, defined on its maximal interval $I_p = (\varpi_-(p), \varpi_+(p))$. If $\varpi_+(p) = \infty$ then the ϖ*-limit set* of p is

$$\varpi(p) = \{q \in \Delta : \exists\{t_n\} \text{ such that } t_n \to \infty \text{ and } \varphi(t_n) \to q \text{ as } n \to \infty\}.$$

In the same way, if $\varpi_-(p) = -\infty$ the α*-limit set* of p is

$$\alpha(p) = \{q \in \Delta : \exists\{t_n\} \text{ such that } t_n \to -\infty \text{ and } \varphi(t_n) \to q \text{ as } n \to \infty\}.$$

Theorem 2. *Assume $c \neq 0$. If $y_1 = 0$ and $z_1 = c$, then system (6) defined on Δ has two invariant hyperplanes $y_2 = 0$ and $y_3 = 0$, and the invariant function $F(y_2, y_3, z_2, z_3, t) = y_2 y_3 \,\mathrm{e}^{2ct}$ (i.e. a first integral depending on the time). Moreover, the following statements hold:*

(a) If $c > 0$, the α-limit of the orbits are in the hyperplanes $y_2 = 0$ or $y_3 = 0$.
(b) If $c < 0$, the ω-limit of the orbits are in the hyperplanes $y_2 = 0$ or $y_3 = 0$.
(c) The function $H(y_2, y_3, z_2, z_3) = (z_2 - z_3)^2 - 2c(z_2 + z_3) + (y_2 - y_3)^2$ is a first integral.

We must mention that under the assumptions of theorem 2 the dynamics on the invariant subspace $y_2 = 0$ is given by the two-dimensional system

$$\dot{y}_3 = y_3(k + z_3), \qquad \dot{z}_3 = -y_3^2,$$

where k is a convenient constant. The orbits of this system are the level curves of its first integral $K = \left(y_3^2 + z_3^2\right)/2 + kz_3$. The dynamics on the subspace $y_3 = 0$ is similar. In short, the α-limits or the ω-limits of the orbits of statements (a) and (b) of theorem 2 are well known.

Since the hyperplane $y_1 = 0$ is the union of the hyperplanes $y_1 = 0$, $z_1 = c$ with $c \in \mathbb{R}$, theorems 1 and 2 provide information on the dynamics over the whole invariant hyperplane $y_1 = 0$.

Our last result states the non-existence of periodic orbits in the hyperplane $y_1 = 0$.

Theorem 3. *System (3) has no periodic orbits in the invariant hyperplane $y_1 = 0$.*

The paper is organized as follows. In section 2 we prove theorems 1 and 2, and in section 3 we prove theorem 3.

2. On the dynamics over the invariant hyperplane $y_1 = 0$

In this section, we will prove theorems 1 and 2.

Proof of theorem 1. Now $c = 0$. We have that $\{(b, b, 0, 0); b \in \mathbb{R}\}$ is a straight line of singular points of system (6). The eigenvalues of the linear part of system (6) at a singular point $(b, b, 0, 0)$ are $0, 0, 2b\mathrm{i}, -2b\mathrm{i}$. To write the linear part of system (6) at $(0, 0, 0, 0)$ into its real Jordan canonical form, we use the following change of coordinates:

$$\begin{aligned} y_2 &= x_2 - x_4, & y_3 &= x_2 + x_4, \\ z_2 &= x_1 - x_3, & z_3 &= x_1 + x_3. \end{aligned} \tag{7}$$

In these coordinates, system (6) with $c = 0$ becomes

$$\begin{aligned} \dot{x}_1 &= -2x_4^2, & \dot{x}_2 &= 2x_3x_4, \\ \dot{x}_3 &= -2x_2x_4, & \dot{x}_4 &= 2x_2x_3. \end{aligned} \tag{8}$$

This system restricted to the variables (x_2, x_3, x_4) is invariant and it is close to the system of the Euler–Lagrange equations of the rigid body with fixed centre of mass. Then its solution is

$$x_2(t) = A\sqrt{-k}\,\mathrm{sn}(2A(t+t_0)|k),$$
$$x_3(t) = A\sqrt{-k}\,\mathrm{cn}(2A(t+t_0)|k),$$
$$x_4(t) = A\,\mathrm{dn}(2A(t+t_0)|k),$$

and by direct integration we get that

$$x_1(t) = B - \frac{E(\mathrm{am}(2A(t+t_0)|k), k)x_4(t)}{\sqrt{1-(x_2(t)/A)^2}},$$

where the functions sn, cn, dn are the Jacobi elliptic functions and E is the elliptic integral of second kind. Of course, A, k, t_0 and B are the constants of integration.

It is easy to check that the three functions

$$F_1(x_1, x_2, x_3, x_4) = x_3^2 + x_4^2 = A^2 k,$$
$$F_2(x_1, x_2, x_3, x_4) = x_2^2 - x_4^2 = -A^2,$$
$$F_3(x_1, x_2, x_3, x_4) = x_1 - \frac{x_4 E\left(\mathrm{am}\left(\mathrm{dn}^{-1}\left(x_4/\sqrt{x_4^2 - x_2^2}|k\right)|k\right)|k\right)}{\sqrt{1 - x_2^2/\left(x_4^2 - x_2^2\right)}}$$

are first integrals of system (8). In this last equality $k = \left(x_3^2 + x_4^2\right)/\left(x_4^2 - x_2^2\right)$. We consider the four 3×3 minors of the 3×4 matrix

$$\partial(F_1, F_2, F_3)/\partial(x_1, x_2, x_3, x_4).$$

Since the intersection where the four minors are zero has Lebesgue measure zero, it follows that F_1, F_2 and F_3 are independent. □

Proof of theorem 2. Now $c \neq 0$. System (6), after the change of coordinates (7), becomes

$$\dot{x}_1 = -2x_4^2, \qquad \dot{x}_2 = 2x_3x_4 - cx_2,$$
$$\dot{x}_3 = -2x_2x_4, \qquad \dot{x}_4 = 2x_2x_3 - cx_4. \tag{9}$$

We observe that $f_1 = x_2 - x_4 = 0$ and $f_2 = x_2 + x_4 = 0$ are invariant hyperplanes. The cofactor of $f_1 = 0$ is $k_1 = -(2x_3 + c)$ and the cofactor of $f_2 = 0$ is $k_2 = 2x_3 - c$. Using the Darboux theory of integrability, see [3, 8], we get that

$$F(x_1, x_2, x_3, x_4, s) = \left(x_2^2 - x_4^2\right)\mathrm{e}^{2ct}$$

is an invariant function of system (9), i.e. $\mathrm{d}F/\mathrm{d}t$ over the orbits of the system is zero.

If $c > 0$ and $t \to -\infty$ then $F \to 0$. It means that $f_1 f_2 \to 0$ and the α-limit of the orbits of system (9) are approaching the hyperplanes $f_1 = 0$ or $f_2 = 0$. So statement (a) of theorem 1 is proved. If $c < 0$ and $t \to \infty$ then $F \to 0$. Again it means that $f_1 f_2 \to 0$ and the ω-limit of the orbits of system (9) are approaching the hyperplanes $f_1 = 0$ or $f_2 = 0$. Hence, this proves statement (b) of theorem 1.

Moreover, if we restrict the first integral H given by (4) to system (6), then we get the first integral of statement (c). This completes the proof of theorem 2. □

3. About the non-existence of periodic orbits

The goal of this section is to prove theorem 3 when $y_1 = 0$ and $z_1 = c =$ constant for all $c \in \mathbb{R}$. The next two propositions establish the result for the cases $c = 0$ and $c \neq 0$, respectively.

Proposition 4. *If $y_1 = 0$ and $z_1 = 0$, then system (3) with $c = 0$ has no periodic orbits.*

Proof. We consider the change of coordinates (7). System (3) restricted to $y_1 = 0$ and $z_1 = 0$ becomes system (6). This system with $c = 0$ after the change of variables (7) becomes system (8). Since $\dot{x}_1 = -2x_4^2 \leqslant 0$, it follows that system (8) has no periodic orbits except if they are contained in $x_4 = 0$. If such a periodic orbit exists, then over it $\dot{x}_2 = 0$ and $\dot{x}_3 = 0$. So, on the periodic orbit x_2 and x_3 are constant. Hence, from the fact that $\dot{x}_4 = 2x_2x_3 =$ constant, such a periodic orbit cannot exists. □

Proposition 5. *If $y_1 = 0$ and $z_1 = c \neq 0$, then system (3) with $c \neq 0$ has no periodic orbits.*

Proof. System (3) restricted to $y_2 = 0$ and $y_3 = c \neq 0$ is

$$\dot{y}_3 = y_3(-z_2 + z_3 - c), \qquad \dot{z}_2 = 0, \qquad \dot{z}_3 = -y_3^2. \tag{10}$$

In order to investigate the existence of periodic orbit of system (3) on the hyperplane $y_1 = 0$, it is sufficient to study the existence for (10). Since $\dot{z}_3 = -y_3^2 \leqslant 0$, the periodic orbits only can exist if $y_3 = 0$, but then it would be formed by singular points, a contradiction. □

Proof of theorem 3. According to statements (a) and (b) of theorem 2, if system (3) restricted to $y_1 = 0$ has a periodic orbit it must be contained into the hyperplanes $y_2 = 0$ or $y_3 = 0$. Therefore, using propositions 6 and 8 we conclude the proof. □

4. Conclusions

We have proved that the Bianchi IX system written into the form (3) has no periodic orbits on the three five-dimensional invariant hyperplanes $y_k = 0$ for $k = 1, 2, 3$, see theorem 3.

For every $k = 1, 2, 3$, we restrict our attention to the four-dimensional invariant hyperplanes $y_k = 0$ and $z_k = c$ with $c \in \mathbb{R}$. If $c = 0$ then the Bianchi IX system restricted to these three four-dimensional invariant hyperplanes is integrable, in the sense that we can explicitly compute their solutions and that we provide three independent first integrals, see theorem 1 and its proof. Moreover, for a given $c \neq 0$ we show that the solutions on these three four-dimensional invariant hyperplanes either start or end in the three-dimensional invariant subhyperplanes planes $y_i = 0$ or $y_j = 0$ with i and j different from k, see theorem 2. Additionally, the flow on these subhyperplanes is easy to study.

Acknowledgments

The first and the third authors are partially supported by a FAPESP-BRAZIL grant 10246-2. The second author is partially supported by a MTM2005-06098-C02-01 and by a CICYT grant no 2005SGR 00550. All authors are supported by the joint project CAPES-MECD grant 071/04 and HBP2003-0017.

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